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Equilibrium existence and uniqueness in impure public good models

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Abstract

Despite widespread application of the impure public good model, surprisingly little attention has been given to the model's equilibrium properties. This paper presents a proof that clearly shows sufficient conditions for existence and uniqueness of a Nash equilibrium.

Keywords: Impure public goods; Equilibrium existence and uniqueness

JEL classification: C72; H41

1. Introduction

Models of privately provided public goods are fundamental in public economics. The standard model is based on a setup in which agents choose between consumption of a private good and contributions to a pure public good. Results are typically based on properties of one or more Nash equilibria, and much attention in the literature has focused on establishing necessary and sufficient conditions for equilibrium existence and uniqueness.[1]

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[1] For examples see Bergstrom, Blume and Varian (1986, 1992), Fraser (1992), Cornes, Hartley and Sandler (1999), and Cornes and Hartley (in press).

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The standard model is a special case of the impure public good model (Cornes and Sandler, 1984, 1994). The notion of an impure public good, which is based on joint production of private and public characteristics, has been adapted to a wide variety of applications. Perhaps the most well-known is the model of warm-glow giving (Andreoni, 1989, 1990). Other applications include the study of military alliances (Sander and Murdoch, 1990), agricultural research (Khanna et al., 1994), household refuse collection (Dubin and Navarro, 1988), pollution abatement (Rubbelke, 2003), and environmentally friendly consumption (Kotchen, 2005).

Despite widespread use of the impure public good model, surprisingly little attention has been given to questions about equilibrium existence and uniqueness. Relying on a proof for the pure public good model, Cornes, Hartley and Sandler (1999) identify bounds on best-response functions that are sufficient for equilibrium existence and uniqueness in the impure public good model. Nevertheless, they explain how the conditions are not intuitive and are of little interest in relation to the typical normality assumption in public goods models. Andreoni (1986) provides a proof for the model of warm-glow giving. While the sufficient conditions are stated in terms of a normality assumption, existence and uniqueness are shown in two different steps, and the analysis ignores the very real possibility for corner solutions in which some individuals do not make a contribution.

Beyond these two studies, there has been no attempt in the literature to provide a general proof of equilibrium existence and uniqueness for impure public good models. Most papers investigate the comparative statics of individual behavior and thereby ignore the game theoretic features of the model. This is a potentially important omission because many of the basic results for pure public goods do not apply for impure public goods. Examples include the facts that neutrality of income redistributions need not hold, Nash behavior does not necessarily result in suboptimal provision, and demand functions need not conform to standard price and income relationships.[2] Thus, it is important that more careful attention be given to equilibrium properties in impure public good models.

This paper provides a general proof of equilibrium existence and uniqueness. The approach extends Cornes and Hartley's (in press) technique for analyzing aggregative public good games. The proof has several advantages. First, it establishes existence and uniqueness in a single, straightforward step that allows for possible corner solutions. Second, the sufficient conditions relate directly to standard normality assumptions. Third, as described by Cornes and Hartley, an inherent advantage of the technique is the nonproliferation of dimensions that comes with the typical approach of using fixed-point theorems. Finally, the proof opens the door for nontrivial extensions of comparative static analysis of individual behavior, as conducted in previous studies, to comparative static analysis of Nash equilibria.

2. The impure public good model

Individuals $i=1, \ldots, n$ are assumed to derive utility from characteristics of goods rather than from goods themselves. There are three characteristics–x, y and Z–where x and y have properties of a private good and Z has properties of a pure public good. Individual preferences are given by a strictly increasing and strictly quasiconcave utility function $U_i=U_i\ (x_i, y_i, Z)$, where $Z=\sum_{i=1}^{n} z_i=Z_{-i}+z_i$.

Characteristics are available through two consumption goods. The first good (a composite numeraire) generates characteristic x_i only, and assuming that the good and characteristic are measured in the same units, the notation x_i can be used to denote both. The second good, denoted q_i, is an impure public good

[2] See Cornes and Sandler (1984, 1994) for further discussion.

that generates y_i and z_i jointly. The technology of joint production is such that one unit of qi generates β>0 units of yi and γ>0 units of z_i.

Each individual's utility maximization problem can be written as

$$\max_{x_i,q_i}\{U_i(x_i,y_i,Z)|\ x_i+pq_i=w_i,\ y_i=\beta q_i,\ Z=Z_{-i}+\gamma q_i\},\tag{1}$$

where p is the price of the impure public good. Specification of this maximization problem for all i individuals establishes the setup of the standard impure public good model (Cornes and Sandler 1984, 1994).

We can further simplify the model by changing the units of q, y, and Z to normalize the exogenous parameters p, β, and γ. First, choose units of the impure public good to normalize its price such that $g_i=pq_i$ denotes the rescaled good. Second, choose units of the jointly produced private characteristic such that one unit of g_i generates one unit of the characteristic, implying that g_i can denote the quantity of the impure public good and its private characteristic. Finally, choose units of the public characteristic such that one unit of g_i generates one unit of it as well, implying that g_i can also denote the quantity of individuals i's provision of the public good. Substituting g_i into (1), and letting $G=\sum_{i=1}^{n} g_i=G_{-i}+g_i$, the utility maximization problem can be rewritten as

$$\max_{x_i,g_i}\{u_i(x_i,g_i,G)|\ x_i+g_i=w_i,\ G=G_{-i}+g_i\},\tag{2}$$

where the rescaled utility function u_i inherits the properties of being strictly increasing and strictly quasiconcave.

Maximization problem (2) is identical to the setup of Andreoni's (1989, 1990) model of warm-glow giving, in which individuals make donations for a combination of 'egoistic' and 'altruistic' reasons. Thus, as the preceding steps make clear, the warm-glow and impure public good models are equivalent; the only difference is the units of the impure public good and its associated characteristics. This equivalence has yet to be shown formally in the literature, and as we will see, it is useful for proving existence and uniqueness of a Nash equilibrium.

It is convenient to continue working with (2) rather than (1). Add G_{-i} to both sides of the budget constraint, substitute it into the objective function, and rewrite the individual's problem with a choice over the aggregate level of the public good:

$$\max_{G\geq G_{-i}} u_i(w_i+G_{-i}-G,G-G_{-i},G).\tag{3}$$

The individual's optimal choice of G will be a continuous function of the exogenous portions of the maximand: $f_i\ (w_i+G_{-i},\ G_{-i})\geq G_{-i}$.[3] Each individual's level of private provision, or best-response function, can then be written as

$$g_i=f_i(w_i+G_{-i},G_{-i})-G_{-i}\geq 0.$$

[3] Andreoni (1989, 1990) defines f_i for only interior solutions, but it causes no trouble to let the function account for corner solutions as well.

This expression shows that each individual either contributes a positive amount or completely free rides and contributes zero.

Typically, in models of privately provided public goods, a further assumption is made that explicitly or implicitly places bounds on best-response functions. Here the assumption is stated as follows:

The normality assumption $0<\partial f_i/\partial G_{-i}\leq 1$.

This requires that an increase in an individual's spillin, G_{-i}, must increase her demand for the public good and not decrease her demand for the private good. Equivalently, it requires that best-response functions have slopes greater than −1 and less than or equal to zero. The assumption is referred to as the normality assumption because the condition is satisfied if we simply assume that both the private and public goods are normal with respect to "full income," which includes personal income plus the value of public good spillins (i.e., $w_i+G_{-\mathrm{sub}\ i}$ in this case). In other words, the assumption is satisfied if preferences are such that the slope of the full-income Engel curve for G is everywhere within (0, 1].[4]

3. Equilibrium existence and uniqueness

Continuing to work with the simplified (warm-glow) formulation, a Nash equilibrium is a set of contributions $\{g_i^*\}_{i=1}^{n}$ that satisfies the aggregation rule $G^*=\sum_{i=1}^{n} g_i^*$. This section proves existence and uniqueness of a Nash equilibrium. The proof is an extension of Cornes and Hartley's (in press) replacement function approach for analyzing aggregative public good games.

Suppress w_i and define the function $h_i\ (G_{-i})=f_i\ (w_i+G_{-i},\ G_{-i})$. Furthermore, let $\underline{G}_i=h_i\ (0)$ for all i, which is the amount of public good that individual i would provide in the absence of any provision by others. We can now define a *replacement function* for each individual and prove its important properties.

Lemma 1. *There exists a well-defined and continuous function, $r_i(G)$, for all i that determines i's contribution level as a function of any $G\geq G_i$, and the function $r_i\ (G)$ satisfies $r_i\ (\underline{G}_i)=\underline{G}_i$ and $r_i'(G)\leq 0$.*[5]

Proof. For $G\geq \underline{G}_i$ define r_i implicitly with $h_i\ (G-r_i)-G\equiv 0$. By the implicit function theorem, $r_i\ (G)$ is a well-defined and continuous function. At interior solutions $r_i>0$, and at corner solutions $r_i=0$. By definition $r_i\ (\underline{G}_i)=\underline{G}_i$ and $r_i'(G) = 1-\frac{1}{h_i'(\cdot)}\leq 0$, where the inequality follows from the i normality assumption. □

Now define $\underline{G}=\max\ \{\underline{G}_i\}_{i=1}^{n}$. With $\underline{G}$ defining the lower bound of its domain, the *aggregate replacement function* and its properties follow directly from the individual replacement functions.

Lemma 2. *There exists a function $R\ (G)=\sum_{i=1}^{n} r_i\ (G)$ that is defined for all $G\geq G$, is continuous, and satisfies $R\ (\underline{G})\geq \underline{G}$ and $R'\ (G)\leq 0$.*

[4] It is straightforward to show, following the same steps, that the normality assumption and its interpretation is the same even in the more general formulation without the normalization of the exogenous parameters.

[5] The function $r_i\ (G)$ is referred to as an individual's replacement function because it is consistent with the following thought experiment: For any exogenous G within the permissible domain, there exists a unique quantity $b\in[0,\ G]$ such that if b were subtracted from G, the individual would choose to replace exactly the quantity removed, so that $b=r_i\ (G)$.

We can use the replacement functions to characterize a Nash equilibrium. For any level of aggregate provision G that might arise, it must be true that $g_i = r_i$ for all i. Hence a Nash equilibrium is a set of replacement values $\{r_i(G^*)\}_{i=1}^{n}$ that satisfies $R(G^*) = G^*$. With this definition, we can prove the main result.

Proposition 1. *There exists a unique Nash Equilibrium.*

Proof. It is sufficient to prove existence of a unique G^* that satisfies $R(G^*) = G^*$. We have shown that $R(G)$ is continuous, nonincreasing, and $R(\underline{G}) \geq \underline{G}$. Hence there exists a solution to $R(G) = G$, and it is unique. □

This proof establishes equilibrium existence and uniqueness in a single step. The approach clearly demonstrates how the normality assumption is sufficient for uniqueness but not necessary for existence. This follows because only continuity of $R(G)$ is necessary for existence, but monotonicity, which is implied by the normality assumption, is sufficient for uniqueness.

4. Conclusion

This paper provides a proof of equilibrium existence and uniqueness for the impure public good model. The approach is to show equivalence between the impure public good model and the warm-glow model and to prove the result for the latter. The proof underscores in a single, straightforward step how the assumptions that are sufficient for existence and uniqueness in the impure public good model are not only identical to those for the warm-glow model, but also those for the pure public good model. The results should serve as a useful reference for future applications and extensions of the impure public good model. As demonstrated in Kotchen (2006), application of the approach for analyzing impure public goods provides a foundation for investigating new results involving comparative static analysis of Nash equilibria. With changes in exogenous conditions, one needs only understand the effects on individual replacement functions in order to understand the effects on equilibrium results.

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